

STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER OPERATOR ON COMPACT LIE GROUPS

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ABSTRACT. Every compact Lie group has a unique covering group which is a product of circles and simply connected semisimple Lie groups. We may equip each component of the covering group a canonical metric, with the requirement that the periods of the geodesic flow on each component are rationally related to each other. Then we push down the metric from the covering group to the original compact Lie group, and call such a metric a “rational metric”. This paper establishes “scaling critical” Strichartz estimates for the Schrödinger operator on compact Lie groups equipped with a rational metric.

1. INTRODUCTION AND BACKGROUNDS

The goal of this paper is to establish the following theorem.

Theorem 1.1. *Let G be a compact Lie group equipped with a “rational metric”, in the sense that it is rational (defined in Theorem 6.1) when passed to its cover $T_1 \times T_n \times K$. Let d be the dimension of G and r the rank of G . Let $F = P_N e^{it\Delta} f_0$, $f_0 \in L^2(G)$ with $\|f_0\|_2 = 1$. Let $m_\lambda = \{(t, x) \in \mathbf{T} \times G : |F(t, x)| > \lambda\}$. Let $p_0 = \frac{2(r+2)}{r}$. Then*

- (i) $m_\lambda \ll N^{\frac{dp_0}{2} - (d+2) + \epsilon} \lambda^{-p_0}$, for $\lambda > CN^{d - \frac{r}{2}}$.
- (ii) $m_\lambda \lesssim N^{\frac{dp}{2} - (d+2)} \lambda^{-p}$, for $\lambda > CN^{d - \frac{r}{2}}$, $p > p_0$.
- (iii)

$$\|P_N e^{it\Delta} f_0\|_p \lesssim N^{\frac{d}{2} - \frac{d+2}{p}}$$

holds for $p \geq 2 + \frac{8}{r}$.

(iv) If the above Strichartz estimate holds for $p > p_0$ with an N^ϵ loss, then it holds for $q > p$ without loss.

The proof of this theorem will combine the known methods of Strichartz estimates on the tori and representation theory and harmonic analysis on semisimple Lie groups and compact Lie groups. As an introduction, we make a short review of known Strichartz estimates for the Schrödinger operator. These include the cases of Euclidean spaces, general Riemannian manifolds with or without boundary, spheres and Zoll manifolds, and tori.

1.1. The case of Euclidean spaces. The Strichartz estimates on the Euclidean space \mathbf{R}^d is as follows.

Theorem 1.2. [20] *Let $f \in L^2(\mathbf{R}^d)$, $d \geq 1$. Then*

$$\|e^{it\Delta} f\|_{L_t^p L_x^q} \lesssim \|f\|_{L_x^2}$$

for admissible pairs (p, q) , i.e. $\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$, $p, q \geq 2$, $(p, q, d) \neq (2, \infty, 2)$. In particular, by Sobolev embedding,

$$\|e^{it\Delta} P_{\leq N} f\|_{L_t^p L_x^q} \lesssim N^{\frac{d}{2} - \frac{2}{p} - \frac{d}{q}} \|f\|_2$$

for $\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}$, $p, q \geq 2$, $(p, q, d) \neq (2, \infty, 2)$.

Note that the admissible condition is natural in the sense of being scale invariant. The geometry of \mathbf{R}^d used in the proof of this theorem is capsulated in the following dispersive estimate

$$(1.3) \quad \|e^{it\Delta} f\|_{\infty} \lesssim |t|^{-d/2} \|f\|_1$$

which is a direct consequence of the explicit formula

$$(1.4) \quad e^{it\Delta} f = \frac{1}{(2\pi t)^{d/2}} \int e^{i|x-y|^2/2t} f(y) dy.$$

Then an direct application of Hardy-Littlewood-Sobolev inequality would give the above theorem except the endpoint case. The Strichartz inequality is a special case of fourier restriction estimate on the parabola, and if we denote by $d\sigma(\xi, -|\xi|^2) = d\xi$, then we can write

$$e^{it\Delta} f = (f d\sigma)^{\vee}$$

and by (1.4)

$$|(d\sigma)^{\vee}| \lesssim |t|^{-d/2}.$$

Note that the same $p = q = \frac{2(d+2)}{d}$ estimate holds for the case that $d\sigma$ is the canonical measure on the sphere by the Tomas-Stein argument, which applied a similar decay estimate

$$|(d\sigma)^{\vee}| \lesssim |x|^{-(d-1)/2}.$$

The endpoint case is then studied in [20], resulting the following abstract Strichartz inequality.

Proposition 1.5. [20] *Let (X, μ) be a σ -finite measured space, and $U : \mathbf{R} \rightarrow B(L^2(X, \mu))$ be a weakly measurable map satisfying*

$$(i) \quad \|U(t)\|_{L^2 \rightarrow L^2} \lesssim 1, t \in \mathbf{R},$$

(ii)

$$\|U(t)U(s)^* f\|_{L^\infty} \lesssim \frac{1}{|t-s|^\sigma} \|f\|_{L^1}, t, s \in \mathbf{R}.$$

Then for every admissible pair p, q one has

$$\|U(t)f\|_{L_t^p L_x^q} \lesssim \|f\|_{L^2}.$$

1.2. The case of general Riemannian manifolds and domains. We have the following Strichartz estimates on compact Riemannian manifolds.

Theorem 1.6. [13] *Let (M, g) be a compact Riemannian manifold of dimension $d \geq 1$ and Δ be the Laplace Beltrami operator on M . Given admissible pairs (p, q) and $q < \infty$, for any finite time interval I , we have*

$$(1.7) \quad \|e^{it\Delta} f\|_{L^p(I, L^q(M))} \lesssim_I \|f\|_{H^s(M)}$$

for $s = \frac{1}{p}$.

The proof hinges on the following dispersion estimate on frequency localized initial data in short time interval comparable with the frequency. While in the case of \mathbf{R}^d , the dispersion is global in time for any initial data.

Lemma 1.8. *[13] Let M be a compact Riemannian manifold of dimension d . Let $\phi \in C_0^\infty(\mathbf{R})$. There exists $\alpha > 0$ and $C > 0$ such that for every $v_0 \in C^\infty(M)$, for every $h \in (0, 1]$*

$$\|e^{it\Delta}\phi(h^2\Delta)v_0\|_{L^\infty(M)} \leq \frac{C}{|t|^{d/2}}\|v_0\|_{L^1(M)}$$

for every $t \in [-\alpha h, \alpha h]$.

This lemma is proved using parametrices for the Schrödinger equation and stationary phase methods. We have a semiclassical heuristics as follows. Let $h \ll 1$. We may imagine such a high energy initial data $\phi(h^2\Delta)v_0$ as particles, moving in the speed of $\sim h^{-1}$ from a point say $x \in M$ in all directions following geodesics, thus disperse at least before they reunite. They will not reunite before they reach the first conjugacy point of x , thus they will not reunite before they traveled a distance of the injectivity radius of M , which is a finite number. Thus we can safely expect dispersion for $|t| \lesssim h^{-1}$.

Then similarly with the case of \mathbf{R}^d , with the help of Proposition 1.2, one has for any interval J of length $|J| \leq \alpha h$,

$$\|e^{it\Delta}\phi(h^2\Delta)v_0\|_{L^p(J, L^q)} \lesssim \|v_0\|_{L^2}.$$

This combined with Littlewood-Paley theory would give the desired estimates. The $1/p$ loss of derivatives comes from the fact that the dispersion is local in time.

The techniques in [13] are robust enough to be applied to prove the same results on many noncompact manifolds. For example, the same results holds for almost all metrics on \mathbf{R}^d .

Theorem 1.9. *[13] Let g be a metric on \mathbf{R}^d satisfying the following uniform bounds*

$$\exists 0 < m < M, \forall x \in \mathbf{R}^d, mId \leq g(x) \leq MId;$$

$$\forall \alpha \in \mathbf{N}^d, \exists C_\alpha, \forall x \in \mathbf{R}^d, |\partial^\alpha g(x)| \leq C_\alpha.$$

Then the same estimates hold as in Theorem 1.3.

Similar heuristics and techniques work for compact domains where the Laplacian is understood as the Dirichlet or Neumann Laplacian. In [1], the author is able to obtain (1.7) for $s = \frac{3}{2p} + \epsilon$ for $d = 2, 3$, which is improved in [4] to $s = \frac{4}{3p}$ for all $d \geq 2$. Also in [5], for two dimensional compact polygonal domains, (1.7) for $s = \frac{1}{p}$ is established, which is the same exponent as in Theorem 1.6.

For some noncompact Riemannian manifolds, Strichartz estimates without loss as for the case of \mathbf{R}^d can be derived. Here an important condition is non-trapping, i.e. geodesics leave a compact set eventually. Established cases include metrics for \mathbf{R}^d which is non-trapping and Euclidean outside a compact set (see [25][7]), asymptotically conic and non-trapping manifolds (see [16]) and asymptotically hyperbolic and non-trapping manifolds (see [6]).

1.3. The case of spheres and Zoll manifolds. To examine the sharpness of the Strichartz estimates on general compact Riemannian manifolds, we may let $f = \phi_\lambda$ be an L^2 normalized eigenfunction of $-\Delta$ with eigenvalue λ , then the Strichartz estimates imply for admissible pairs (p, q)

$$(1.10) \quad \|\phi_\lambda\|_{L^q} \lesssim \lambda^{1/p}.$$

We also have the following estimates on spectral projectors $\Pi_k := \chi_{[k^2, (k+1)^2]}(-\Delta)$ on any compact Riemannian manifold.

Theorem 1.11. [23]

$$\|\Pi_k\|_{L^2 \rightarrow L^q} \lesssim k^{s(q)}$$

$$\text{where } s(q) = \begin{cases} \frac{d-1}{2}(\frac{1}{2} - \frac{1}{q}), & \text{if } 2 \leq q \leq \frac{2(d+1)}{d-1}, \\ \frac{d-1}{2} - \frac{d}{q}, & \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty. \end{cases}$$

In particular, this theorem imply that

$$(1.12) \quad \|\phi_\lambda\|_{L^q} \lesssim \lambda^{s(q)}.$$

Comparing (1.10) and (1.12), one sees that for $d \geq 3$, $p = 2$, $q = \frac{2d}{d-2}$, (1.10) and (1.12) match; for $d = 2$, let p tend to 2, (1.10) is worse than (1.12) by just a λ^ϵ loss. Since Theorem 1.11 is sharp, for example optimal for $M = S^d$, $d \geq 2$ by testing against spherical harmonics (for other examples see [24]), thus the Strichartz estimates is also sharp for $d \geq 3$, $p = 2$, $q = \frac{2d}{d-2}$, and for $d = 2$, p tending 2 with an ϵ loss, sharpness obtained for example by $M = S^d$, $d \geq 2$.

For S^d , $d \geq 2$, we also have the following Strichartz estimates. In fact, the following estimates hold generally for a class of compact manifolds – Zoll manifolds, which are compact manifolds such that all geodesics are closed with common periods. Note that for $d \geq 3$, the estimates are “scale invariant” and also hold for \mathbf{R}^d and \mathbf{T}^d .

Theorem 1.13. [13][14][17] *Let M be a Zoll manifold of dimension $d \geq 2$. Let I be any finite time interval and let $p > 4$. Then for $d \geq 3$,*

$$\|P_N e^{it\Delta} f\|_{L^p(I \times M)} \lesssim N^{\frac{d}{2} - \frac{d+2}{p}} \|f\|_{L^2(M)}.$$

For $d = 2$,

$$\|P_N e^{it\Delta} f\|_{L^p(I \times M)} \lesssim \begin{cases} N^{\frac{3}{4} - \frac{5}{2p}} \|f\|_{L^2(M)}, & \text{if } 4 < p \leq 6, \\ N^{1 - \frac{4}{p}} \|f\|_{L^2(M)}, & \text{if } 6 \leq p \leq \infty. \end{cases}$$

In particular, the above estimates hold for $p = 4$ with an ϵ of loss.

The proof has several ingredients. First, the only geometry of Zoll manifold used in deriving the above estimate is about the distribution of eigenvalues of $-\Delta$.

Proposition 1.14. [13] *If the geodesics of M are 2π periodic, there exists $\alpha \in \mathbf{N}$ and $A > 0$ such that the spectrum of $-\Delta$ is contained in $\cup_{k=1}^\infty I_k$, where $I_k = [(k + \frac{\alpha}{4})^2 - A, (k + \frac{\alpha}{4})^2 + A]$.*

Next, let P_k be the spectral projector on I_k . If we define $\tilde{\Delta}$ to be $-\sum_{k=1}^\infty (k + \frac{\alpha}{4})^2 P_k$, then borrowing Bourgain’s Fourier restriction spaces, one sees that Strichartz estimates for $e^{it\Delta}$ are equivalent to those for $e^{it\tilde{\Delta}}$ (See [14]). Note that P_k enjoys the same estimates as Π_k in Theorem 1.11.

The last essential ingredients is the following estimates on exponential sums, proof relying on the circle method.

Proposition 1.15. *[8][17] Let $p > 4$ and $\alpha \in \mathbf{N}$. It holds that*

$$\left\| \sum_{n \in \mathbf{Z} \cap J} c_n e^{it\mu_n^2} \right\|_{L_t^p([0, 32\pi])} \lesssim N^{1/2-2/p} \left(\sum_{n \in \mathbf{Z} \cap J} |c_n|^2 \right)^{\frac{1}{2}}$$

for every $J = [b, b + N]$ and $\mu_n = n + \alpha/4$.

Then we may put all the ingredients together as done in Lemma 3.5 in [17]. Let's copy it here. Assume $d \geq 3$ while $d = 2$ can be done similarly. Write

$$P_N e^{it\tilde{\Delta}} f(x) = \sum_{n \sim N} e^{-it\mu_n^2} P_n f(x),$$

then Proposition 1.15 implies that

$$\|P_N e^{it\tilde{\Delta}} f(x)\|_{L_t^p} \lesssim N^{1/2-2/p} \left(\sum_{n \sim N} |P_n f(x)|^2 \right)^{\frac{1}{2}}.$$

Then take the L^p norm in x , applying Minkowski's inequality and Sogge's estimates as in Theorem 1.11, we have

$$\left\| \left(\sum_{n \sim N} |P_n f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} \lesssim \left(\sum_{n \sim N} \|P_n f\|_{L^p}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{n \sim N} N^{2(\frac{d-1}{2}-\frac{d}{p})} \|P_n f\|_{L^2}^2 \right)^{\frac{1}{2}} \lesssim N^{\frac{d-1}{2}-\frac{d}{p}} \|f\|_{L^2}$$

which implies the desired Strichartz estimates. For sharpness, one can also show that for $p = 4$, the ϵ loss is necessary for $S^d, d \geq 3$, and $\|P_N e^{it\Delta} f\|_{L^4(I \times S^2)} \lesssim N^\delta \|f\|_{L^2(S^2)}$ can not hold for $\delta < \frac{1}{8}$.

1.4. The case of tori. There is a longer story for the case of torus in the pursuing of the following theorem.

Theorem 1.16. *[9][11][21] Let \mathbf{T}^d be a torus (not necessarily square) of dimension d . Then for any finite time interval I ,*

$$\|P_N e^{it\Delta} f\|_{L^p(I \times \mathbf{T}^d)} \lesssim_\epsilon N^{\frac{d}{2}-\frac{d+2}{p}+\epsilon} \|f\|_{L^2(\mathbf{T}^d)}$$

for $p > \frac{2(d+2)}{d}$. Furthermore, if \mathbf{T}^d is a rectangular one – i.e. when the metric is $g = \otimes_{i=1}^d c_i^2 dt_i^2$ where c_i 's are nonzero reals and dt_i^2 the canonical metric on each circle component, then the above estimates are improved to be without the ϵ loss.

The Strichartz estimates on tori belong to the general discrete restriction phenomena.

1.4.1. Dispersive estimates and Tomas-Stein type arguments. Let's focus on the square torus first. In [9], $p = 6$ for $d = 1$ and $p = 4$ for $d \geq 2$ is first established with an ϵ loss by reducing the problem into counting integral points on circles or algebraic curves. To get estimates without loss and with lower p for high dimensions, a more analytical method reminiscent of the Tomas-Stein restriction theorem and the circle method is devised. Here are the key ingredients of the argument. Just as in the Euclidean case, Fourier analysis helps one to write out $P_N e^{it\Delta}$ as a convolution operator with kernel

$$K(t, x) = \sum_{\xi} e^{-it|\xi|^2 + ix \cdot \xi} \Phi\left(\frac{\xi}{N}\right) = \prod_{i=1}^d \sum_{\xi_i} e^{-it\xi_i^2 + ix_i \xi_i} \phi\left(\frac{\xi_i}{N}\right).$$

The torus being a compact Riemannian manifold, we expect dispersion of the type in (1.8) for $|t| \leq \frac{C}{N}$

$$\|P_N e^{it\Delta} f\|_{L^\infty(\mathbf{T}^d)} \lesssim |t|^{-d/2} \|f\|_{L^1(\mathbf{T}^d)}.$$

Combining with Bernstein's inequality

$$\|P_N e^{it\Delta} f\|_{L^\infty(\mathbf{T}^d)} \lesssim N^d \|f\|_{L^1(\mathbf{T}^d)}$$

we have for $|t| \leq C/N$,

$$\|P_N e^{it\Delta} f\|_{L^\infty(\mathbf{T}^d)} \lesssim \left(\frac{N}{1 + N|t|^{1/2}}\right)^d \|f\|_{L^1(\mathbf{T}^d)}.$$

Now the special feature of the geometry of the torus is that the Schrödinger flow enjoys similar dispersion with good bounds near rational times $\frac{t}{2\pi} = \frac{a}{q}$ for $q < N$. This is due to the Weyl type exponential sum estimates of K .

Proposition 1.17. [9]

$$\|P_N e^{it\Delta} f\|_{L^\infty(\mathbf{T}^d)} \lesssim \left(\frac{N}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})}\right)^d \|f\|_{L^1(\mathbf{T}^d)}$$

for $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$, $(a, q) = 1$, $q < N$.

Borrowing terminology of the circle method, these dispersion estimates motivate the decomposition of the time interval $[0, 2\pi]$ into major arcs and minor arcs (Q, M dyadic)

$$1 = \sum_{1 \leq Q \leq \frac{N}{100}} \sum_{Q \leq M \leq N} \left[\left(\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q} \right) * \omega_{\frac{1}{NM}} \right] + \rho(t)$$

and accordingly

$$K = \sum_{1 \leq Q \leq \frac{N}{100}} \sum_{Q \leq M \leq N} \left[\left(\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q} \right) * \omega_{\frac{1}{NM}} \right] K + \rho(t)K.$$

From Proposition 1.17, we have the $L^1 \rightarrow L^\infty$ norm estimates of both major and minor arc components of K .

The next ingredient is the $L^2 \rightarrow L^2$ norm estimates (to obtain an estimate without loss, an $L^1 + L^2 \rightarrow L^2$ estimate is also needed) of the major arc components of K . Using Fourier analysis, this can be done by estimating the L^∞ norm of the space-time Fourier transform of major components of K , which in turn reduces to estimating the Fourier transform of $\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q}$. This then relates to the number of divisors which have classical bounds that can be applied.

Interpolating $L^1 \rightarrow L^\infty$ and $L^2 \rightarrow L^2$, we get the following sharp distributional Strichartz estimates.

Proposition 1.18. [9] Let $\|f\|_2 = 1$. Then we have

$$\begin{aligned} (1) & \|P_N e^{it\Delta} f\| > \lambda \ll N^\epsilon \lambda^{-\frac{2(d+2)}{d}} \text{ for } \lambda > N^{\frac{d}{4}} \\ (2) & \|P_N e^{it\Delta} f\| > \lambda \lesssim N^{\frac{d}{2}p - (d+2)} \lambda^{-p} \text{ for } \lambda > N^{\frac{d}{4}}, p < \frac{2(d+2)}{d}. \end{aligned}$$

A consequence of this proposition is to remove any N^ϵ factor in Strichartz estimates, thus the estimates with N^ϵ loss mentioned earlier by the arithmetical method of counting of lattice points on curves can be updated to get the full estimates for $d = 1, 2$. For higher dimensions, interpolating (2) with the trivial subcritical estimate

$$\|P_N e^{it\Delta} f\|_2 \lesssim \|f\|_2$$

would give the result for $p \geq \frac{2(d+4)}{d}, d \geq 4$.

1.4.2. Multilinear restriction estimates and Bourgain-Guth induction on scales. To get critical estimates for smaller exponents, one could first prove subcritical estimates for larger exponents. In fact, by interpolating Proposition 1.18 with subcritical estimates, we see that the larger the exponent in the subcritical estimate, the smaller the exponent in the critical estimates. And of course proving subcritical estimates is in itself worthwhile. This observation is the start of [10], where for the subcritical estimates the exponent is proved to be as large as $p = 2(d+1)/d$, which implies the critical estimate for exponents as small as $p = 2(d+3)/d$.

The foundation of [10] is the following multilinear restriction estimate from [2].

Theorem 1.19. [2] *Let $\tau_1, \dots, \tau_{d+1}$ be transverse caps on the parabola $\{(\xi, |\xi|^2) : |\xi| \leq 1\}$ and assumes \hat{f}_i is supported on the $1/N$ -neighborhood of τ_i . Then we have*

$$\|(\Pi_{i=1}^{d+1} |f_i|)^{1/(d+1)}\|_{L^{\frac{2(d+1)}{d}}(B_N)} \ll N^{-\frac{1}{2}+\epsilon} (\Pi_{i=1}^{d+1} \|f_i\|_{L^2})^{1/(d+1)}.$$

By some localization arguments (for details one can see [27]), this also implies the following multilinear discrete restriction estimate

$$\|(\Pi_{i=1}^{d+1} |e^{it\Delta} f_i|)^{1/(d+1)}\|_{L^{\frac{2(d+1)}{d}}} \ll N^\epsilon (\Pi_{i=1}^{d+1} \|f_i\|_{L^2})^{1/(d+1)}$$

for the f_i 's living on the transverse caps τ_i 's respectively.

If not for the transverse condition, letting all $f_i = f$, one would get the desired subcritical estimate for $p = \frac{2(d+1)}{d}$. In order to overcome this barrier, first in [12] J. Bourgain and L. Guth developed an induction on scales method that could retrieve linear restriction estimates from multilinear restriction estimate (not necessarily of the same exponents). Then in [10] a similar method is applied to get the critical Strichartz estimate for $p = \frac{2(d+1)}{d}$ from the above multilinear discrete restriction estimates.

Following arguments of similar spirit, finally in [11], the full range of Strichartz estimates are obtained with an epsilon loss for both square and irrational tori. Now Proposition 1.18 can be used to remove this epsilon loss for square torus, and the for the irrational torus, similar methods are developed in [21].

In addition to the linear Strichartz estimates, there are bilinear or multilinear refinements. These are motivated to prove local well-posedness for nonlinear Schrödinger equations, both for subcritical and critical data, with the help of Bourgain's Fourier restriction spaces and U_p, V_p spaces respectively. For examples see [9][18][19][17][21][15].

2. PRELIMINARIES

This section gives an overview of fundamental results on representation theory and harmonic analysis on semisimple Lie groups and compact Lie groups, necessary to the Strichartz estimates for the Schrödinger operator.

2.1. Peter-Weyl theorem and Hausdorff-Young inequality.

Theorem 2.1 (Peter-Weyl). *Let \hat{G} be the set of the equivalent classes of irreducible unitary representations of the compact group G , let (V_j, π_j) be the irreducible unitary representation in the class j , and let $\mathcal{M}_j := \{\text{tr}(A\pi_j(x)) : A \in \text{End}(V_j)\}$, the space of matrix coefficients w.r.t j . Then $\bigoplus_{j \in \hat{G}} \mathcal{M}_j$ is dense in $L^2(G)$. Moreover,*

let $d\mu$ be the normalized Haar measure, let $d_j = \dim(V_j)$, then for $f \in L^2(G)$, we have the L^2 -convergent Fourier series

$$f(x) = \sum_{j \in \hat{G}} d_j \operatorname{tr}(\hat{f}(j) \pi_j(x)), \quad \hat{f}(j) = \int_G f(x) \pi_j(x^{-1}) d\mu.$$

Moreover, we have Schur's orthogonality conditions

$$\int_G \operatorname{tr}(A \pi_j(x)) \overline{\operatorname{tr}(B \pi_k(x))} d\mu = \frac{\delta_{jk}}{d_j} \operatorname{tr}(AB^*),$$

from which we get

$$\|f\|_{L^2(G)} = \left(\sum_{j \in \hat{G}} d_j \|\hat{f}(j)\|_{HS}^2 \right)^{1/2}, \quad \langle f, g \rangle_{L^2(G)} = \sum_{j \in \hat{G}} d_j \operatorname{tr}(\hat{f}(j) \hat{g}(j)^*).$$

Here $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm of matrices.

Define the convolution

$$(f * g)(x) = \int_G f(xy^{-1}) g(y) d\mu(y).$$

Then

$$(f * g)^\wedge(j) = \hat{f}(j) \hat{g}(j).$$

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \quad r, p, q \geq 1.$$

Also if $\hat{g}(j) = \lambda_j \cdot \operatorname{Id}_{d_j \times d_j}$, λ_j a scalar, then

$$(2.2) \quad \|f * g\|_{L^2(G)} \leq \sup_j |\lambda_j| \cdot \|f\|_{L^2(G)}.$$

We have a Hausdorff-Young inequality for compact groups G (see [22]):

$$(2.3) \quad \left(\sum_{j \in \hat{G}} d_j \|\hat{f}(j)\|_{S_{p'}}^{p'} \right)^{1/p'} \leq \|f\|_{L^p(G)}$$

where $1 \leq p \leq 2$, $\|\hat{f}(j)\|_{S_{p'}} = (\operatorname{tr}|\hat{f}(j)|^{p'})^{1/p'}$, $|\hat{f}(j)| := \sqrt{\hat{f}(j) \hat{f}(j)^*}$, with the exception that when $p' = \infty$, $\|\hat{f}(j)\|_{S_\infty}$ is the largest eigenvalue of $|\hat{f}(j)|$, and the left hand side of (2.3) is understood as $\sup_{j \in \hat{G}} \|\hat{f}(j)\|_{S_\infty}$. In particular,

$$(2.4) \quad \|\hat{f}(j)\|_{HS} \leq d_j^{1/2} \|f\|_{L^1(G)}, \quad \forall j \in \hat{G}.$$

2.2. Classification of compact Lie groups.

Theorem 2.5. [28] *Let G be a compact Lie group and \mathfrak{g} be its Lie algebra. Then \mathfrak{g} is reductive, i.e. \mathfrak{g} is a direct sum of its center and $\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, with $\mathfrak{D}\mathfrak{g}$ being semisimple. And $\mathfrak{D}\mathfrak{g}$ is of compact type, i.e. its adjoint group is compact.*

Theorem 2.6 (Lie). [28] *Let \mathfrak{g} be any Lie algebra over \mathbf{R} (or \mathbf{C}). Then there exists a unique simply connected Lie group whose Lie algebra is isomorphic to \mathfrak{g} . We may call this group the analytic group of \mathfrak{g} .*

Theorem 2.7 (Weyl). [28] *Let \mathfrak{g} be semisimple over \mathbf{R} and of compact type. Then the analytic group of \mathfrak{g} is compact.*

Combining these theorems, one sees that the universal cover of a compact Lie group G is $\mathbf{R}^d \times K$, where K is compact simply connected and semisimple. Since the image is \mathbf{R}^d is necessarily \mathbf{T}^d , the universal covering maps factor through $\mathbf{T}^d \times K$. Thus we have the exact sequence

$$1 \rightarrow A \rightarrow \tilde{G} \cong \mathbf{T}^d \times K \rightarrow G \rightarrow 1.$$

A is finite and normal thus lies in the center of \tilde{K} . Then by complexification, there is one to one correspondence between isomorphism classes of compact simply connected and semisimple Lie groups and complex semisimple Lie algebras, which are completely classified by E. Cartan.

2.3. Structure of compact semisimple Lie groups, Weyl's character formula, theorem of the highest weight. Let G be a compact semisimple Lie group and \mathfrak{g} its Lie algebra. Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$. Choose a maximal torus $B \subset G$. Let \mathfrak{b} be the Lie algebra of B and $\mathfrak{b}_{\mathbf{C}}$ its complexification. As a torus, the Fourier dual $\hat{B} \cong \Lambda \subset i\mathfrak{b}^*$, Λ is often called the weight lattice. B is unique up to conjugation and conjugations of B cover G .

The adjoint action of B on $\mathfrak{g}_{\mathbf{C}}$ are all semisimple (can be made unitary) and commute, thus diagonalizable simultaneously, which gives a decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{g}_{\mathbf{C}}^{\alpha})$. Here $\Phi \subset \Lambda - \{0\}$ is the set of roots, $\mathfrak{g}_{\mathbf{C}}^{\alpha} = \{X \in \mathfrak{g}_{\mathbf{C}} : \text{Ad}_b(X) = e^{\alpha(b)}X \text{ for all } b \in B\}$, which has dimension 1.

The Weyl group $W =: N(B)/B$ is finite and acts faithfully on $B, \mathfrak{b}_{\mathbf{C}}, i\mathfrak{b}^*, \Lambda, \Phi$, and leaves invariant an inner product – the Cartan-Killing form on $i\mathfrak{b}^*$.

Facts about the Weyl group W :

- (1) W , as group acting on $i\mathfrak{b}^*$, is generated by reflections about hyperplanes α^{\perp} , $\alpha \in \Phi$;
- (2) W permutes the connected components of $i\mathfrak{b}^* - \cup_{\alpha \in \Phi} \alpha^{\perp}$ – the Weyl chambers – simply transitively.

Fix a particular Weyl chamber C^+ , define $P = \{\alpha \in \Phi : \langle \alpha, C^+ \rangle \subset \mathbf{R}_{>0}\}$, the set of positive roots. Then Φ is a disjoint union of P and $-P$.

For every set of positive roots P , there corresponds to a unique subset S , called the set of simple roots, which is defined to be linearly independent and every positive root is a positive integer linear combination of simple roots.

Let $S = \{\alpha_1, \dots, \alpha_r\}$. Define $\{w_1, \dots, w_r\} \subset i\mathfrak{b}^*$ such that $\langle w_i, \frac{2\alpha_j}{|\alpha_j|^2} \rangle$, that is, $\{w_1, \dots, w_r\}$ is the dual basis w.r.t. $\{\frac{2\alpha_1}{|\alpha_1|^2}, \dots, \frac{2\alpha_r}{|\alpha_r|^2}\}$. $\{w_1, \dots, w_r\}$ generate the so-called integral lattice L . Facts about the integral lattice L :

- (1) The Weyl chamber $C^+ = \{\sum_{i=1}^r x_i w_i : x_i > 0, 1 \leq i \leq r\}$.
- (2) $i\mathfrak{b}^* - \cup_{\alpha \in \Phi} \alpha^{\perp} = \{\sum_{i=1}^r x_i w_i : x_i \neq 0, 1 \leq i \leq r\}$.
- (3) $\Lambda \subset L$. If G is simply-connected, then $\Lambda = L$; vice versa.
- (4) $\rho := \frac{1}{2} \sum_{\alpha \in P} \alpha = \sum_{i=1}^r w_i$.

In particular, these facts give that for a compact simply-connected semisimple Lie group, $\{\lambda \in \Lambda : \Pi_{\alpha} \langle \alpha, \lambda \rangle \neq 0\} = \cup_{s \in W} s((\Lambda + \rho) \cap C^+)$.

Theorem 2.8 (Weyl). *[28] $\hat{G} \cong (\Lambda + \rho) \cap C^+$, with $\lambda \in (\Lambda + \rho) \cap C^+$ corresponding to the class $j \in \hat{G}$ with character given by the formula*

$$\chi_j|_B = \frac{\sum_{w \in W} \epsilon(w) e^{w\lambda}}{\sum_{w \in W} \epsilon(w) e^{w\rho}} = \frac{\sum_{w \in W} \epsilon(w) e^{w\lambda}}{e^{-\rho} \prod_{\alpha \in P} (e^{\alpha} - 1)}$$

where $\epsilon(w) = \det\{w : i\mathfrak{b}^* \rightarrow i\mathfrak{b}^*\} \subset \{\pm 1\}$. We also have the formula of the dimension for $j \in \hat{G}$

$$d_j = \frac{\Pi_{\alpha \in P} \langle \alpha, \lambda \rangle}{\Pi_{\alpha \in P} \langle \alpha, \rho \rangle}.$$

Let $H \in \mathfrak{b}$. We can think of $-iH$ as a real linear functional on $i\mathfrak{b}^*$, and by the Cartan-Killing inner product on $i\mathfrak{b}^*$, we thus get a correspondence between $H \in \mathfrak{b}$ and an element in $i\mathfrak{b}^*$. Thus $e^{\lambda(H)} = e^{i\langle \lambda, H \rangle}$ and we may rewrite the Weyl character formula as follows

$$\chi_j(\exp H) = \frac{\sum_{w \in W} \epsilon(w) e^{i\langle w\lambda, H \rangle}}{\sum_{w \in W} \epsilon(w) e^{i\langle w\rho, H \rangle}} = \frac{\sum_{w \in W} \epsilon(w) e^{i\langle w\lambda, H \rangle}}{e^{-i\langle \rho, H \rangle} \Pi_{\alpha \in P} (e^{i\langle \alpha, H \rangle} - 1)}.$$

Lemma 2.9. [28] Let $\{w_1, \dots, w_r\}$ be fundamental weights of Λ , $\Lambda = \{\sum_{i=1}^r n_i w_i : n_i \in \mathbf{Z}, 1 \leq i \leq r\}$. Then χ_j , viewed as a function of (n_1, \dots, n_r) , satisfies $(D_i f := f(\dots, n_i + 1, \dots) - f(\dots, n_i, \dots))$

$$(2.10) \quad |D_{i_1} \cdots D_{i_n} \chi_j(n_1, \dots, n_r)| \lesssim N^{p-n}$$

for $|\sum_{i=1}^r n_i w_i| \lesssim N$, $\|\langle \alpha, H \rangle\| \ll N^{-1}$, $\|\cdot\|$ the distance to 0 on a $[0, 2\pi)$ torus, p the number of positive roots.

Suppose that the irreducible representation (π_j, V_j) corresponds to $\lambda \in (\Lambda + \rho) \cap C^+$ as in the statement of Weyl character formula. Since B is compact abelian, $V_j = \oplus_{\mu \in \Lambda} V_j^\mu$, with $V_j^\mu = \{v \in V_j : \pi(b)(v) = e^\mu(b)v \text{ for all } b \in B\}$. This is the so-called weight decomposition of V_j . We call μ a weight of π_j if $V_j^\mu \neq 0$.

Theorem 2.11. Any irreducible representation π_j admits a highest weight μ , which is defined by either of the following two equivalent statements:

- (1) μ is a weight of π_j , but $\mu + \alpha$ is not for any $\alpha \in P$;
- (2) μ is a weight of π_j , and any other weight of π_j can be expressed as μ minus a sum of positive roots.

When this is the case, $\dim V_j^\mu = 1$, and μ determines π uniquely among irreducible representations of G . In the notation of the Weyl character formula, $\mu = \lambda - \rho$.

2.4. Spectra of the Laplacian. The canonical choice of Riemannian metric for compact semisimple Lie group G is the negative of Cartan-Killing form. The Cartan-Killing form is defined as

$$\langle X, Y \rangle = \text{tr}(\text{ad} X \text{ad} Y), \quad X, Y \in \mathfrak{g}.$$

It is negative definite on \mathfrak{g} and invariant the under adjoint action of G . Thus the corresponding Riemannian metric is bi-invariant.

Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification of \mathfrak{g} . Let $\mathfrak{B}_{\mathbf{C}}$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$, i.e. the tensor algebra of $\mathfrak{g}_{\mathbf{C}}$ modulo the ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$, $X, Y \in \mathfrak{g}_{\mathbf{C}}$. Then for $Y_1, \dots, Y_n \in \mathfrak{g}_{\mathbf{C}}$, the expression $Y_1 \cdots Y_n$ can be understood as an element of $\mathfrak{B}_{\mathbf{C}}$ or as a left invariant differential operator on G . These two perspectives give an isomorphism between $\mathfrak{B}_{\mathbf{C}}$ and the space of left invariant differential operators.

Under this isomorphism, bi-invariant differential operators correspond to the center of $\mathfrak{B}_{\mathbf{C}}$. Given an analytic representation (π, V) of G , it induces a representation of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{B}_{\mathbf{C}}$. One see that right regular representation on the space of the matrix coefficients decomposes into $d = \dim(V)$ copies of (π, V) , and the

induced action of $\mathfrak{B}_{\mathbf{C}}$ on V corresponds to the left invariant differentiation on matrix coefficients. If a lies in the center of $\mathfrak{B}_{\mathbf{C}}$ and (π, V) irreducible, then Schur's lemma tells us that the induced action of a on V is a constant times identity, thus the bi-invariant differentiation of the matrix coefficients is also this constant times identity.

The Laplacian-Beltrami operator $\Delta = \sum_{i=1}^n X_i^2$, where $\{X_1, \dots, X_n\}$ is an orthonormal basis of \mathfrak{g} , is bi-invariant and as an element of $\mathfrak{B}_{\mathbf{C}}$ (called the Casimir operator) lies in its center. The definition does not depend on choices of basis, in fact for any two basis $\{Y_1, \dots, Y_n\}$ and $\{Y^1, \dots, Y^n\}$ of \mathfrak{g} dual to each other w.r.t. the Riemannian metric, one has $\Delta = \sum_{i=1}^n Y_i Y^i$.

Let (π_j, V_j) be an irreducible representation of G corresponding to a highest weight μ . By the above discussion, Δ acts on matrix coefficients as a constant multiple of identity. To compute this constant, we test the Casimir operator on v_j , a nonzero vector in V_j^μ . We have the root space decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}} \oplus_{\alpha \in P} (\mathbf{C}e_\alpha \oplus \mathbf{C}f_\alpha)$. Let e_α, f_α be chosen such that $\langle e_\alpha, f_\alpha \rangle = 1$.

Now let $\{k_i, 1 \leq i \leq l\}$ be an orthonormal basis of $\mathfrak{b}_{\mathbf{C}}$ w.r.t. the Cartan-Killing form. Then $\{e_\alpha, f_\alpha, \alpha \in P; k_i, 1 \leq i \leq l\}$ is dual to $\{f_\alpha, e_\alpha, \alpha \in P; k_i, 1 \leq i \leq l\}$, thus the Laplacian viewed as the Casimir operator can be expressed as $-\Delta = \sum_{\alpha \in P} e_\alpha f_\alpha + f_\alpha e_\alpha + \sum_i k_i^2$. Combine

- (1) $e_\alpha v_j = 0$ since $e_\alpha(V_j^\mu) = V_j^{\mu+\alpha}$ and μ is the highest weight;
- (2) $e_\alpha f_\alpha v_j = [e_\alpha, f_\alpha]v_j = \langle e_\alpha, f_\alpha \rangle \langle \alpha, \mu \rangle v_j$;
- (3) $\sum_i k_i^2 v_j = \sum_i \langle \mu, k_i \rangle^2 v_j = \langle \mu, \mu \rangle v_j$, we get

$$-\Delta v_j = (\langle \mu + \rho, \mu + \rho \rangle - \langle \rho, \rho \rangle) v_j.$$

Thus the matrix coefficients of (π_j, V_j) corresponding to highest weight μ are eigenfunctions of $-\Delta$ with eigenvalue $\langle \mu + \rho, \mu + \rho \rangle - \langle \rho, \rho \rangle$. Since all matrix coefficients generate $L^2(G)$, the spectra computation is complete.

2.5. $\mathrm{SU}(n)$ as an example. We have the following results for $G = \mathrm{SU}(n)$. Let $\mathfrak{su}(n)$ be its Lie algebra. From now on, we use λ rather than j to denote an element of \hat{G} .

- (1) The biinvariant metric (Cartan-Killing form) on $\mathfrak{su}(n)$ is $\langle X, Y \rangle := -\mathrm{tr}(X^*Y)$.

A Cartan subalgebra \mathfrak{t} is given by $\mathfrak{t} = \left\{ \begin{pmatrix} i\theta_1 & & \\ & \ddots & \\ & & i\theta_n \end{pmatrix} \middle| \theta_j \in \mathbb{R}, \sum_{i=1}^n \theta_i = 0 \right\}$.

The roots are $\alpha_{kl} := E_{kk} - E_{ll}$ where E_{ij} is the matrix with only a nonzero ij -th entry with value 1. $\{\alpha_{kl} \mid k < l\}$ form a positive system, $\{\alpha_{kl} \mid k > l\}$ form a negative system, and $\{\alpha_{k,k+1}\}$ form a system of simple roots. The Weyl group W is all permutations on the diagonal entries of \mathfrak{t} , thus is isomorphic to the permutation group S_n . The dominant integral elements are

$$\left\{ \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \middle| m_j \in \mathbb{R}, \sum_{i=1}^n m_i = 0, m_i - m_{i+1} \text{ are nonnegative integers for all } i \right\}.$$

This set can be reparametrized by $n - 1$ integers

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq 0$$

with

$$m_i = \lambda_i - \frac{\lambda}{n}, \quad 1 \leq i \leq n-1,$$

$$m_n = -\frac{\lambda}{n}, \quad \lambda = \sum_{i=1}^{n-1} \lambda_i.$$

This also gives a parametrization of \hat{G} .

(2)

$$k_\lambda = \sum_{i=1}^n (m_i + \rho_i)^2 - \rho_i^2$$

where

$$\rho_i = \frac{1}{2} \sum_{\alpha \in P} \alpha_i = \frac{n+1}{2} - i.$$

(3)

$$d_\lambda = \frac{\prod_{i < j} (m_i - m_j + j - i)}{\prod_{i < j} (j - i)}.$$

(4)

$$\chi_\lambda(t) = \frac{A_{m+\rho}(t)}{A_\rho(t)},$$

$$\text{where } t \in T = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \middle| \theta_j \in \mathbb{R}, \sum_{i=1}^n \theta_i = 0 \right\} =$$

$\exp t$, $A_\alpha(t)$ is the Vandermonde determinant of the matrix $(t_i^{\alpha_j})$.

Specialize to $G = SU(2)$, we have the following. $\lambda \in \hat{G}$ is parametrized by nonnegative integers $m \geq 0$,

$$d_m = m + 1,$$

$$k_m = m(m + 2),$$

$$\chi_m \left(\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} \right) = \sum_{j=0}^m e^{i(m-2j)\theta} = \frac{e^{i(m+1)\theta} - e^{-i(m+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin(m+1)\theta}{\sin \theta}.$$

For a central function $f(\theta)$,

$$(2.12) \quad \int_{SU(2)} f \, d\mu = 2 \int_0^{2\pi} f(\theta) \sin^2 \theta \frac{d\theta}{2\pi}.$$

Specialize to $G = SU(3)$, we have the following. $\lambda \in \hat{G}$ is parametrized by N_1, N_2 two positive integers, letting $\lambda_2 = N_2 - 1$, $\lambda_1 - \lambda_2 = N_1 - 1$.

$$k_{N_1, N_2} = \left(\frac{2N_1 + N_2}{3} \right)^2 + \left(\frac{2N_2 + N_1}{3} \right)^2 + \left(\frac{N_1 - N_2}{3} \right)^2 - 2 = \frac{2}{3} (N_1^2 + N_2^2 + N_1 N_2) - 2.$$

$$d_{N_1, N_2} = \frac{N_1 N_2 (N_1 + N_2)}{2}.$$

$$\begin{aligned}
 \chi_{N_1, N_2}(\theta_1, \theta_2) &= \chi_{N_1, N_2} \left(\begin{pmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{-i\theta_1 - i\theta_2} \end{pmatrix} \right) \\
 &= \frac{e^{i(N_1\theta_1 + N_2(\theta_1 + \theta_2))} - e^{i(N_1\theta_2 + N_2(\theta_1 + \theta_2))} - e^{i(N_1\theta_1 - N_2\theta_2)} \\
 &\quad - e^{-i(N_1(\theta_1 + \theta_2) + N_2\theta_1)} + e^{i(N_1\theta_2 - N_2\theta_1)} + e^{-i(N_1(\theta_1 + \theta_2) + N_2\theta_2)}}{e^{i(2\theta_1 + \theta_2)} - e^{i(2\theta_2 + \theta_1)} - e^{i(\theta_1 - \theta_2)} \\
 &\quad - e^{-i(2\theta_1 + \theta_2)} + e^{i(\theta_2 - \theta_1)} + e^{-i(\theta_1 + 2\theta_2)}} \\
 &= \frac{e^{i((N_1 + N_2)\theta_1 + N_2\theta_2)} - e^{i((N_1 + N_2)\theta_2 + N_2\theta_1)} - e^{i(N_1\theta_1 - N_2\theta_2)} \\
 &\quad - e^{-i((N_1 + N_2)\theta_1 + N_1\theta_2)} + e^{-i((N_1 + N_2)\theta_2 + N_1\theta_1)} + e^{i(N_1\theta_2 - N_2\theta_1)}}{e^{i(2\theta_1 + \theta_2)} - e^{i(2\theta_2 + \theta_1)} - e^{i(\theta_1 - \theta_2)} \\
 &\quad - e^{-i(2\theta_1 + \theta_2)} + e^{-i(\theta_1 + 2\theta_2)} + e^{i(\theta_2 - \theta_1)}}
 \end{aligned}$$

Letting

$$2\theta_1 + \theta_2 = t_1, 2\theta_2 + \theta_1 = t_2,$$

we may rewrite the above as

$$\chi_{N_1, N_2}(t_1, t_2) = \frac{e^{i\frac{2N_1 + N_2}{3}t_1 + i\frac{N_2 - N_1}{3}t_2} - e^{i\frac{2N_1 + N_2}{3}t_2 + i\frac{N_2 - N_1}{3}t_1} - e^{i\frac{2N_1 + N_2}{3}t_1 - i\frac{2N_2 + N_1}{3}t_2} \\
 - e^{-i\frac{2N_2 + N_1}{3}t_1 + i\frac{N_2 - N_1}{3}t_2} + e^{-i\frac{2N_2 + N_1}{3}t_2 + i\frac{N_2 - N_1}{3}t_1} + e^{-i\frac{2N_2 + N_1}{3}t_1 + i\frac{2N_1 + N_2}{3}t_2}}{e^{it_1} - e^{-it_1} - e^{it_2} + e^{-it_2} - e^{i(t_1 - t_2)} + e^{-i(t_1 - t_2)}}.$$

3. THE SCHRÖDINGER KERNEL

Write $f(x) = \sum_{j \in \hat{G}} d_j \text{tr}(\pi_j(x) \hat{f}(j))$, we have

$$P_N e^{it\Delta} f = \sum_{j \in \hat{G}} \phi\left(\frac{k_j}{N^2}\right) e^{-itk_j} d_j \text{tr}(\pi_j(x) \hat{f}(j)),$$

where $k_j = \langle \mu_j + \rho, \mu_j + \rho \rangle - \langle \rho, \rho \rangle$ with μ_j being the highest weight corresponding to $j \in \hat{G}$. This implies $(P_N e^{it\Delta} f)^\wedge(j) = \phi\left(\frac{k_j}{N^2}\right) e^{-itk_j} \hat{f}(j)$. Define

$$(K_N(t, \cdot))^\wedge(j) = \phi\left(\frac{k_j}{N^2}\right) e^{-itk_j} \text{Id}_{d_j \times d_j},$$

$$(3.1) \quad K_N(t, x) = \sum_{j \in \hat{G}} \phi\left(\frac{k_j}{N^2}\right) e^{-itk_j} d_j \chi_j(x)$$

where $\chi_j = \text{tr}(\pi_j)$ the characters. Then

$$P_N e^{it\Delta} f = K_N(t, \cdot) * f = f * K_N(t, \cdot).$$

Now we are interested in obtaining estimates for a finite time interval I

$$\|P_N e^{it\Delta} f\|_{L^p(I, L^q(G))} \lesssim N^s \|f\|_{L^2(G)}.$$

Lemma 3.2. [28] *For a compact semisimple Lie group G and its weight lattice Λ , there exists $D \in \mathbf{N}$ such that $\langle \lambda_1, \lambda_2 \rangle \in D^{-1}\mathbf{Z}$ for any $\lambda_1, \lambda_2 \in \Lambda$.*

Thus $k_j = \langle \mu_j + \rho, \mu_j + \rho \rangle - \langle \rho, \rho \rangle \in D^{-1}\mathbf{Z}$, $e^{it\Delta} f$ is periodic with a period of $2\pi D$ and we can think of t as living on the torus $\mathbf{T} = [0, 2\pi D)$. Now we fix the time interval I to be \mathbf{T} . We can also think of the Strichartz estimates as Fourier restriction estimates on the product group $\mathbf{T} \times G$.

Remember that $(\mathbf{T} \times G)^\wedge = D^{-1}\mathbf{Z} \times \hat{G}$, we have the following computation of space-time Fourier transforms. For $(n, j) \in D^{-1}\mathbf{Z} \times \hat{G}$

$$\hat{K}_N(n, j) = \begin{cases} \phi(k_j/N^2) \cdot \text{Id}_{d_j \times d_j}, & \text{if } n = -k_j, \\ 0 & \text{otherwise.} \end{cases}$$

For $m(t) = \sum_{n \in D^{-1}\mathbf{Z}} \hat{m}(n) e^{itn}$, we compute that

$$(3.3) \quad (mK_N)^\wedge(n, j) = \hat{m}(n + k_j) \phi\left(\frac{k_j}{N^2}\right) \text{Id}_{d_j \times d_j}.$$

The argument of TT^* makes equivalent of the above Strichartz estimate with

$$\|\tilde{K}_N * F\|_{L^p(\mathbf{T}, (L^q(G)))} \lesssim N^{2s} \|F\|_{L^{p'}(\mathbf{T}, L^{q'}(G))}$$

with $\tilde{K}_N(t, \cdot) = K_N(t, \cdot) * K_N(t, \cdot) = \sum_{j \in \hat{G}} \phi^2(\frac{k_j}{N^2}) e^{-itk_j} d_j \chi_j$. We will not distinguish K_N and \tilde{K}_N in the following.

Specialize to $G = SU(2)$, from (3.1), we have

$$(3.4) \quad K(t, \theta) = \sum_{m \geq 0} \phi\left(\frac{m(m+2)}{N^2}\right) (m+1) e^{-im(m+2)t} \frac{\sin(m+1)\theta}{\sin \theta}.$$

Let $n = m + 1$, we have

$$(3.5) \quad K(t, \theta) \sim \frac{e^{it}}{2} \sum_{n \in \mathbf{Z}} \phi\left(\frac{n}{N}\right) n \cdot \frac{\sin n\theta}{\sin \theta} \cdot e^{in\theta - in^2 t}.$$

We also have

$$(3.6) \quad K(t, \theta) = \frac{e^{it}}{2i \sin \theta} \sum_{n \in \mathbf{Z}} \phi\left(\frac{n}{N}\right) n e^{in\theta - in^2 t}.$$

Note that the function ϕ in the above two equations are slightly different.

4. THE TOMAS-STEIN TYPE ARGUMENTS FOR STRICHARTZ ESTIMATES ON $SU(2)$

To tackle the Strichartz estimates on general compact Lie groups, we first look at the simplest nonabelian case, that is of $SU(2)$. $SU(2)$ is isomorphic to S^3 as Riemannian manifolds, thus by the Strichartz estimates on Zoll manifolds, we have Theorem 1.13. Now we'd like to adapt the Tomas-Stein type argument in [9] for the tori, trying to retrieve Theorem 1.13 for S^3 . This method will then be generalized to cover all compact Lie groups with a “rational metric”. We will establish critical distributional Strichartz estimates for large λ for all $p > 6$ and critical Strichartz estimates for all $p \geq 10$.

First, let's derive dispersive estimates. We have the following result on Weyl type exponential sums.

Lemma 4.1. [9] *Let $f(t, \theta) = \sum_{n \in \mathbf{Z}} \phi(\frac{n}{N}) g(n) e^{i(n\theta - n^2 t)}$, $N \in \mathbf{N}$, where $g(n)$ is any function that satisfies*

$$|D^m g|(n) \lesssim N^{l-m}, l \in \mathbf{N}, \forall m \in \mathbf{N}.$$

Here $(Dg)(n) := g(n+1) - g(n)$. Let $(a, q) = 1$ such that $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$ and $q < N$. Then

$$|f(t, \theta)| \lesssim \frac{N^{l+1}}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})}.$$

Using this lemma for (3.6), (3.5), we have for $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$

$$(4.2) \quad K(t, \theta) \leq \frac{1}{\sin \theta} \cdot \frac{N^2}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})}, \text{ for all } \theta \in [0, 2\pi),$$

$$(4.3) \quad K(t, \theta) \leq \frac{N^3}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})}, \text{ for } |\sin \theta| \lesssim \frac{1}{N}.$$

In particular, for $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$,

$$(4.4) \quad \|K(t, \cdot)\|_\infty \lesssim \frac{N^3}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})}.$$

Following [9], for $N_1 = \frac{N}{100}$, write

$$(4.5) \quad 1 = \sum_{1 \leq Q \leq N_1} \sum_{Q \leq M \leq N} [(\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q}) * \omega_{\frac{1}{NM}}] + \rho(t).$$

Then if $\rho(t) \neq 0$, then by Dirichlet's lemma on rational approximations, $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$, $q > N_1 = \frac{N}{100}$, which combined with (4.4) implies

$$(4.6) \quad \|\rho(t)K(t, \cdot)\|_\infty \lesssim N^{5/2}.$$

Define coefficients $\alpha_{Q,M}$ such that

$$(4.7) \quad [(\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q}) * \omega_{\frac{1}{NM}}]^\wedge(0) = \alpha_{Q,M} \hat{\rho}(0)$$

then we have

$$(4.8) \quad \alpha_{Q,M} \lesssim \frac{Q^2}{NM}.$$

Then write

$$\begin{aligned} K(t, x) &= \sum_{Q \leq N_1} \sum_{Q \leq M \leq N} K(t, x) [((\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q}) * \omega_{\frac{1}{NM}}) - \alpha_{Q,M} \rho](t) \\ &\quad + (1 + \sum_{Q,M} \alpha_{Q,M}) K(t, x) \rho(t). \end{aligned}$$

Define

$$(4.9) \quad \Lambda_{Q,M}(t, x) = K(t, x) [((\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q}) * \omega_{\frac{1}{NM}}) - \alpha_{Q,M} \rho](t).$$

Then from (4.4), (4.6), (4.8), we have

$$(4.10) \quad |\Lambda_{Q,M}| \lesssim N^2 \left(\frac{NM}{Q}\right)^{1/2}$$

Next, estimate $\hat{\Lambda}_{Q,M}$. From (3.3)

$$(4.11) \quad \hat{\Lambda}_{Q,M}(n, m) = \lambda_{Q,M}(n, m) \cdot \text{Id}_{(m+1) \times (m+1)}$$

where

$$(4.12) \quad \lambda_{Q,M}(n, m) = \phi\left(\frac{m(m+2)}{N^2}\right) [(\sum_{(a,q)=1, Q \leq q < 2Q} \delta_{a/q})^\wedge \cdot \hat{\omega}_{\frac{1}{NM}} - \alpha_{Q,M} \hat{\rho}](n + m(m+2)).$$

Using bound

$$(4.13) \quad |(\sum_{(a,q)=1, q \sim Q} \delta_{a/q})^\wedge(n)| \ll d(n, Q)Q^{1+\epsilon}, n \neq 0$$

we get

$$(4.14) \quad |\lambda_{Q,M}(n, m)| \ll (\frac{Q^{1+\epsilon}}{NM}d(n+m(m+2), Q) + \frac{Q^2}{NM}|\hat{\rho}(n+m(m+2))|).$$

From (4.13) and (4.5), we have for $n \neq 0$, $|n| \leq CN^2$,

$$(4.15) \quad |\hat{\rho}(n)| \leq \sum_{Q,M} \frac{d(n, Q)Q^{1+\epsilon}}{NM} \ll \frac{N^\epsilon}{N}$$

thus

$$(4.16) \quad |\lambda_{Q,M}(n, m)| \ll \phi(\frac{m}{N}) \frac{Q}{NM} [Q^\epsilon d(n+m(m+2), Q) + \frac{Q}{N^{1-\epsilon}}]$$

$$(4.17) \quad \ll \phi(\frac{m}{N}) \frac{Q^{1+\epsilon}}{NM}.$$

We assume here $|n| \lesssim N^2$.

From (4.10) one gets

$$(4.18) \quad \|f * \Lambda_{Q,M}\|_{L^\infty(\mathbf{T} \times G)} \leq \|f\|_1 \|\Lambda_{Q,M}\|_\infty \lesssim N^2 (\frac{NM}{Q})^{1/2} \|f\|_1.$$

From now on, we need to assume that $\hat{f}(n, m) = 0$ for $n > CN^2$. From (2.2), (4.11) and (4.16), one gets

$$\|f * \Lambda_{Q,M}\|_{L^2(\mathbf{T} \times G)} \ll \frac{Q}{NM} N^\epsilon \|f\|_2.$$

More precisely,

$$(4.19) \quad \|f * \Lambda_{Q,M}\|_2 \lesssim \frac{Q^{1+\epsilon}}{NM} [\sum_{n,m} \phi(\frac{m}{N})^2 (m+1) \|\hat{f}(n, m)\|_{HS}^2 d(n+m(m+2), Q)^2]^{1/2} + \frac{Q^2}{MN^{2-\epsilon}} \|f\|_2$$

$$(4.20) \quad \lesssim \frac{Q^{1+\epsilon}}{NM} [\sum_{n,m} \phi(\frac{m}{N})^2 ((m+1)^{-1/2} \|\hat{f}(n, m)\|_{HS})^2 (m+1)^2 d(n+m(m+2), Q)^2]^{1/2}$$

$$(4.21) \quad + \frac{Q^2}{MN^{2-\epsilon}} \|f\|_2.$$

We have

$$(4.22) \quad \#\{(n, m) : 0 \leq m \lesssim N, d(n+m(m+2), Q) > D\} \leq C_{\tau,B} (D^{-B} Q^\tau N^2 + Q^B) N.$$

(2.4) gives

$$(m+1)^{-1/2} \|\hat{f}(n, m)\|_{HS} \lesssim \|f\|_1,$$

then we get

$$(4.23) \quad \|f * \Lambda_{Q,M}\|_2 \lesssim (\frac{Q^{1+\epsilon} D}{NM} + \frac{Q^2}{MN^{2-\epsilon}}) \|f\|_2 + \frac{Q}{NM} \cdot Q \cdot C_{\tau,B} (D^{-B/2} Q^\tau N + Q^{B/2}) N^{3/2} \|f\|_1.$$

Take $L > 1, D = LQ^\tau$ and assume

$$(4.24) \quad B > \frac{6}{\tau}, N > (LQ)^B$$

then we get

$$(4.25) \quad \|f * \Lambda_{Q,M}\|_2 \lesssim \frac{Q^{1+2\tau}L}{NM} \|f\|_2 + C_{\tau,B} M^{-1} L^{-B/2} N^{3/2} \|f\|_1.$$

Now interpolating (4.18) and (4.19), we get

$$(4.26) \quad \|f * \Lambda_{Q,M}\|_{L^p} \ll N^{2-\frac{4}{p}+\epsilon} (NM)^{\frac{1}{2}-\frac{3}{p}} Q^{-(\frac{1}{2}-\frac{3}{p})} \|f\|_{L^{p'}}.$$

Interpolating (4.18) and (4.25), for

$$(4.27) \quad p > 6 + 4\tau, \sigma = \frac{1}{2} - \frac{3+2\tau}{p} > 0$$

$$(4.28)$$

$$\|f * \Lambda_{Q,M}\|_p \lesssim Q^{-\sigma} L^{2/p} N^{2-\frac{4}{p}} (NM)^{\frac{1}{2}-\frac{3}{p}} \|f\|_p + C_{\tau,B} Q^{-1/4} M^{\frac{1}{2}-\frac{3}{p}} L^{-\frac{B}{p}} N^{3/2} \|f\|_1$$

Now let $F = P_N e^{it\Delta} f_0$, $\|f_0\|_2 = 1$. Then $\|F\|_2 \lesssim 1$. Let's prove the following proposition.

Proposition 4.29. *Let $m_\lambda = |(t, x) \in \mathbf{T} \times G : |F(t, x)| > \lambda|$. Then*

(i) $m_\lambda \ll N^{4+\epsilon} \lambda^{-6}$ for $\lambda > CN^{5/4}$.

(ii) $m_\lambda \lesssim N^{\frac{3p}{2}-5} \lambda^{-p}$ for $\lambda > CN^{5/4}$, $p > 6$.

Proof. Let $f = \chi_{|F|>\lambda} \cdot \frac{F}{|F|}$. Then

$$\begin{aligned} \lambda m_\lambda &\leq |\langle F, f \rangle| = |\langle F * K, f \rangle| = |\langle F * K, Q_{N^2} f \rangle| = |\langle F, Q_{N^2} f * K \rangle| \\ &\leq \|F\|_2 \|Q_{N^2} f * K\|_2 \lesssim |\langle Q_{N^2} f, Q_{N^2} f * K * K \rangle|^{1/2} = |\langle Q_{N^2} f, Q_{N^2} f * \tilde{K} \rangle|^{1/2}. \end{aligned}$$

Here $\tilde{K} = K * K$ which enjoys the same established bounds as K , thus we will replace \tilde{K} by K abusing notation. Q_{N^2} is the spectral projection on the scale N^2 , which is to accommodate the need for $\hat{f}(n, m) = 0$ for $n > CN^2$ to establish the bounds on $f * \Lambda_{Q,M}$ previously. Note that $\|Q_{N^2} f\|_p \lesssim \|f\|_p$, we will replace $Q_{N^2} f$ by f abusing notation. Write

$$(4.30) \quad \Lambda = \sum_{Q,M} \Lambda_{Q,M}, K = \Lambda + (K - \Lambda)$$

we have

$$(4.31) \quad \lambda^2 m_\lambda^2 \lesssim |\langle f, f * \Lambda \rangle| + |\langle f, f * (K - \Lambda) \rangle|$$

$$(4.32) \quad \leq \|f\|_{p'} \|f * \Lambda\|_p + \|f\|_1^2 \|K - \Lambda\|_\infty.$$

Using (4.26) for $p = 6$, summing over Q, M , the first term of 4.31 is bounded by

$$(4.33) \quad N^{\frac{4}{3}+\epsilon} \|f\|_{6'}^2 \leq N^{\frac{4}{3}+\epsilon} m_\lambda^{2/6'}.$$

From (4.6) and (4.8) we get

$$(4.34) \quad \|K - \Lambda\|_\infty \lesssim N^{5/2}.$$

The second term of (4.31) is thus bounded by

$$(4.35) \quad N^{5/2} \|f\|_1^2 \lesssim N^{5/2} m_\lambda^2.$$

Then we have

$$(4.36) \quad \lambda^2 m_\lambda^2 \lesssim N^{\frac{4}{3}+\epsilon} m_\lambda^{2/6'} + N^{5/2} m_\lambda^2$$

which implies for $\lambda > CN^{5/4}$

$$(4.37) \quad m_\lambda \lesssim N^{4+\epsilon} \lambda^{-6}.$$

Thus (i) is proved. To prove (ii) for fixed p , from (i) it suffices to prove it for $\lambda > N^{\frac{3}{2}-\epsilon}$. Summing (4.28) over Q, M in the range indicated by (4.24), we get

$$(4.38) \quad \|f * \Lambda_1\|_p \lesssim LN^{3-\frac{10}{p}} \|f\|_{p'} + L^{-B/p} N^{2-\frac{3}{p}} \|f\|_1$$

denoting

$$(4.39) \quad \Lambda_1 = \sum_{Q < Q_1, M} \Lambda_{Q, M}$$

where Q_1 is the largest Q -value satisfying (4.24). For values $Q \geq Q_1$, use (4.26) to get

$$(4.40) \quad \|f * (\Lambda - \Lambda_1)\|_p \ll N^{3-\frac{10}{p}+\epsilon} Q_1^{-(\frac{1}{2}-\frac{3}{p})} \|f\|_{p'}.$$

Using (4.31) and proceeding as before, we get

$$(4.41) \quad \lambda^2 m_\lambda^2 \lesssim N^{3-\frac{10}{p}} (L + \frac{N^\epsilon}{Q_1^{\frac{1}{2}-\frac{3}{p}}}) m_\lambda^{2/p'} + L^{-B/p} N^{2-\frac{3}{p}} m_\lambda^{1+\frac{1}{p'}} + N^{5/2} m_\lambda^2.$$

For $\lambda > CN^{5/4}$, the last term of the above inequality may be dropped. Let $Q_1 = N^\delta$ such that $\delta > 0$ has to satisfy (4.24)

$$(4.42) \quad (LN^\delta)^B < N.$$

Thus L dominate $\frac{N^\epsilon}{Q_1^{\frac{1}{2}-\frac{3}{p}}}$ for $p > 6 + 4\tau$, thus

$$(4.43) \quad \lambda^2 m_\lambda^2 \lesssim N^{3-\frac{10}{p}} L m_\lambda^{2/p'} + L^{-B/p} N^{2-\frac{3}{p}} m_\lambda^{1+\frac{1}{p'}}.$$

This implies

$$(4.44) \quad m_\lambda \lesssim N^{-5} (N^{3/2}/\lambda)^p L^{p/2} + N^{-p-3} (N^{3/2}/\lambda)^{2p} L^{-B}.$$

Make

$$(4.45) \quad L = (\frac{N^{3/2}}{\lambda})^\tau, B > \frac{p}{\tau}$$

and sufficiently small δ to satisfy

$$(4.46) \quad (LN^\delta)^B < N \Leftarrow (N^\epsilon N^\delta)^B < N$$

to finally get

$$(4.47) \quad m_\lambda \lesssim N^{-5} (N^{3/2}/\lambda)^{p+\frac{p\tau}{2}}.$$

Note that conditions for p, τ indicated in (4.27) implies that $p + \frac{p\tau}{2}$ can take any exponent > 6 . This completes the proof of (ii). \square

Corollary 4.48. (i)

$$(4.49) \quad \|P_N e^{it\Delta} f\|_p \lesssim N^{\frac{3}{2}-\frac{5}{p}} \|f\|_2$$

holds for $p \geq 10$.

(ii) If (4.49) holds for $p_0 > 6$ with an N^ϵ loss, then it holds for $p > p_0$ without loss.

The proof of this corollary is the same with Proposition 3.110 and 3.113 in [9]. It suffices to note that the range of λ in Proposition 4.29 is $[0, CN^{3/2}]$ due to the following Bernstein's inequality (which holds for all compact Riemannian manifolds, see [13])

$$(4.50) \quad \|P_N e^{it\Delta} f\|_\infty \lesssim N^{3/2} \|f\|_2.$$

Remark 4.51. In the above Tomas-Stein type argument, to exploit the oscillation of

$$(4.52) \quad \tilde{\Lambda}_{Q,M} = K(t, x) \left(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} * \omega_{\frac{1}{NM}} \right)(t)$$

one looks at its Fourier transform

$$(4.53) \quad (\tilde{\Lambda}_{Q,M})^\wedge(n, m) = \phi\left(\frac{m(m+2)}{N^2}\right) \left[\left(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} \right)^\wedge \cdot \hat{\omega}_{\frac{1}{NM}} \right](n + m(m+2)) \cdot \text{Id}_{(m+1) \times (m+1)}$$

and try to estimate its decay. Since $(\sum_{(a,q)=1, q \sim Q} \delta_{a/q})^\wedge(n)$ enjoys much better decay (4.13) when $n \neq 0$ than $n = 0$, thus we introduce (4.7) and use $\Lambda_{Q,M}$ in (4.9) instead of $\tilde{\Lambda}_{Q,M}$ so to avoid the case $n = 0$. Now I will apply the arguments in [21] with some modifications to reprove the ϵ removal for $p > 6$, i.e. Corollary (4.48) (ii). This argument deals with the space-time convolution $f * K$ by doing space convolution estimates first then temporal convolution estimates, in similarity with the treatment of non-endpoint cases of Strichartz estimates on Euclidean space, thus avoids doing the space-time Fourier transform. This approach would inevitably have to treat the positive temporal kernel $(\sum_{(a,q)=1, q \sim Q} \delta_{a/q}) * \omega_{\frac{1}{NM}}$ directly, and the method of changing it to $(\sum_{(a,q)=1, q \sim Q} \delta_{a/q}) * \omega_{\frac{1}{NM}} - \alpha_{Q,M} \rho$ would no longer work in this case. But for $L^p \rightarrow L^q$ estimates on convolution with a positive kernel it is still possible to exploit its oscillation by its Fourier transform.

Let

$$(4.54) \quad \Lambda_0 = \sum_{Q < N^\sigma, Q \leq M \leq N} \tilde{\Lambda}_{Q,M}.$$

$\tilde{\Lambda}_{Q,M}$ as in (4.52). Let's prove the following.

Proposition 4.55. *Let $p > 4, r > 6$ or $\frac{1}{r} + \frac{2}{p} < \frac{1}{2}$. $G = SU(2)$. Then*

$$(4.56) \quad \|f * \Lambda_0\|_{L_t^p L_x^r(\mathbf{T} \times G)} \lesssim N^{3 - \frac{4}{p} - \frac{6}{r}} \|f\|_{L_t^{p'} L_x^{r'}(\mathbf{T} \times G)}$$

for $\sigma > 0$ sufficiently small depending on (p, r) only.

Let's see how this proposition implies the desired ϵ removal for exponents bigger than 6. By (4.4) and Dirichlet's lemma, we have

$$(4.57) \quad \|K - \Lambda_0\|_\infty \lesssim N^{3 - \frac{\sigma}{2}}.$$

Replacing Λ by Λ_0 in (4.31), applying the above bound and Proposition (4.55) for $p = r > 6$, we get

$$(4.58) \quad \lambda^2 m_\lambda^2 \lesssim N^{3 - \frac{10}{p}} m_\lambda^{2/p'} + N^{3 - \frac{\sigma}{2}} m_\lambda^2$$

which implies for $\lambda > CN^{\frac{3}{2} - \frac{\sigma}{4}}$

$$(4.59) \quad m_\lambda \lesssim N^{\frac{3p}{2} - 5} \lambda^{-p}$$

for $\sigma > 0$ sufficiently small depending on $p > 6$. Then the epsilon removal would follow by the same argument that established Proposition 3.113 in [9] and Corollary 4.48 above.

To prove Proposition 4.55, we first deal with spatial convolution with $K(t, \cdot)$. Let $f \in L^{r'}(G)$. For $(a, q) = 1$, $|\frac{t}{2\pi} - \frac{a}{q}| \leq \frac{1}{qN}$, we have for $2 \leq r \leq \infty$

$$(4.60) \quad \|f * K(t, \cdot)\|_r \lesssim \left(\frac{N^3}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \right)^{1 - \frac{2}{r}} \|f\|_{r'}.$$

by interpolating between

$$(4.61) \quad \|f * K(t, \cdot)\|_\infty \leq \|K(t, \cdot)\|_\infty \|f\|_1 \lesssim \frac{N^3}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \|f\|_1$$

and

$$(4.62) \quad \|f * K(t, \cdot)\|_2 \lesssim \|f\|_2.$$

Then we have

$$(4.63) \quad \|f * \Lambda_0\|_r \lesssim \sum_{Q < N^\sigma, Q \leq M \leq N} N^{\frac{5}{2} - \frac{5}{r}} M^{\frac{1}{2} - \frac{1}{r}} Q^{-\frac{1}{2} + \frac{1}{r}} \left(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} \right) * \omega_{\frac{1}{NM}} \|f\|_{r'}.$$

For temporal convolution, we have the following slight modification of Lemma 2.5 in [21].

Lemma 4.64. Fix $2 < p \leq \infty$. Then for any $\sigma < \min\{\frac{p}{2} - 1, 1\}$,

$$(4.65) \quad \|[(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} * \omega_{\frac{1}{NM}}] * f\|_{L^p(\mathbf{T})} \lesssim Q^{\frac{2}{p}(1+\epsilon)} (NM)^{-\frac{2}{p}} \|f\|_{L^{p'}(\mathbf{T})}$$

for $\epsilon = \frac{\sigma^2 + 3\sigma}{1 - \sigma}$ uniformly for $1 \leq Q \leq N^\sigma$ and $Q \leq M \leq N$.

By Young's inequality, we easily get

$$\|[(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} * \omega_{\frac{1}{NM}}] * f\|_{L^p(\mathbf{T})} \lesssim Q^{\frac{4}{p}} (NM)^{-\frac{2}{p}} \|f\|_{L^{p'}(\mathbf{T})}.$$

This trivial estimate does not exploit the oscillation of the temporal kernel. It could contribute to some Strichartz estimates but with larger exponents compared using the above lemma. This is similar with the classical Tomas-Stein argument where failure to exploit oscillation but only the decay of the kernel $(d\sigma)^\wedge$ gives worse restriction estimates. See [26].

Combine the above lemma with (4.63), we have for $F \in L_t^p L_x^r$

$$(4.66) \quad \|F * \Lambda_0\|_{p,r} \lesssim \sum_{Q \leq N^\sigma, Q \leq M \leq N} N^{\frac{5}{2} - \frac{5}{r} - \frac{2}{p}} M^{\frac{1}{2} - \frac{1}{r} - \frac{2}{p}} Q^{-\frac{1}{2} + \frac{1}{r} + \frac{2}{p}(1+\epsilon)} \|F\|_{p',r'}$$

$$(4.67) \quad \lesssim N^{3 - \frac{4}{p} - \frac{6}{r}} \|F\|_{p',r'}$$

for $\frac{1}{r} + \frac{2}{p} < \frac{1}{2}$, $\epsilon = \frac{\sigma^2 + 3\sigma}{1 - \sigma} < \frac{p}{2}(\frac{1}{2} - \frac{1}{r} - \frac{2}{p})$. Now one can slightly ungrade (4.60) in the range of $r > 6$ into

$$(4.68) \quad \|f * K(t, \cdot)\|_r \lesssim \frac{N^{3 - \frac{6}{r}}}{\sqrt{q}(1 + N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \|f\|_{r'}.$$

By Young's inequality, this could result from

$$(4.69) \quad \|K(t, \cdot)\|_{\frac{p}{2}} \lesssim \frac{N^{3-\frac{6}{p}}}{\sqrt{q}(1+N|\frac{t}{2\pi}-\frac{a}{q}|^{1/2})}$$

which can be deduced from Weyl's integral formula (2.12) as follows. For $3 < p < \infty$, using (4.2) and (4.3),

$$\begin{aligned} \|K(t, \cdot)\|_{L^p(SU(2))}^p &= \int_0^{2\pi} |K(t, \theta)|^p \cdot 2 \sin^2 \theta \frac{d\theta}{2\pi} \\ &= \int_{|\sin \theta| \leq \frac{1}{N}} + \int_{|\sin \theta| > \frac{1}{N}} \\ &\lesssim \left(\frac{N^3}{\sqrt{q}(1+N|\frac{t}{2\pi}-\frac{a}{q}|^{1/2})} \right)^p N^{-3} + \left(\frac{N^2}{\sqrt{q}(1+N|\frac{t}{2\pi}-\frac{a}{q}|^{1/2})} \right)^p N^{p-3} \\ &\lesssim \left(\frac{N^{3-\frac{3}{p}}}{\sqrt{q}(1+N|\frac{t}{2\pi}-\frac{a}{q}|^{1/2})} \right)^p. \end{aligned}$$

This completes the proof of Proposition 4.55.

5. THE CASE OF $SU(2)^d \times \mathbf{T}^e$

Now we'd like to consider the case of products $SU(2)^d \times \mathbf{T}^e$, i.e. d copies of $SU(2)$ and e copies of \mathbf{T} . We assume the Riemannian metric is defined as

$$(5.1) \quad g = \otimes_{i=1}^{d+e} \alpha_i^2 g_i$$

where g_1, \dots, g_d are the canonical metric for $SU(2)$ and g_{d+1}, \dots, g_{d+e} are the canonical metric for \mathbf{T} : for both metrics, the linear Schrödinger flow is 2π -periodic, and the α_i are positive scalars. Let's distinguish between a rational product and an irrational product. A rational product is defined such that all α_i 's are rational multiples of a certain number, while an irrational is not such. From now on, let's assume the product being rational. We will write out the Tomas-Stein arguments for the "square" case, that is when all α_i 's are 1, and indicate the necessary modification for the general rational case.

First for two compact Lie groups $G_1 \times G_2$, we have $(G_1 \times G_2)^\wedge = \hat{G}_1 \times \hat{G}_2$. For $\lambda_i \in \hat{G}_i$, $i = 1, 2$, we have

$$(5.2) \quad d_{\lambda_1, \lambda_2} = d_{\lambda_1} d_{\lambda_2},$$

$$(5.3) \quad k_{\lambda_1, \lambda_2} = k_{\lambda_1} + k_{\lambda_2},$$

$$(5.4) \quad \chi_{\lambda_1, \lambda_2} = \chi_{\lambda_1} \otimes \chi_{\lambda_2}.$$

From these and the Schrödinger kernel expression (3.1), we get

$$(5.5) \quad K(t, x_1, x_2) = K(t, x_1) \cdot K(t, x_2).$$

Thus the Schrödinger kernel for the square product $SU(2)^d \times \mathbf{T}^e$ is

(5.6)

$$K(t, \theta_1, \dots, \theta_{d+e}) = \prod_{i=1}^d \left(\sum_{m_i \geq 0} \phi\left(\frac{m_i(m_i+2)}{N^2}\right) (m_i+1) e^{-im_i(m_i+2)t} \frac{\sin(m_i+1)\theta_i}{\sin \theta_i} \right) \\ \cdot \prod_{i=d+1}^{d+e} \left(\sum_{n_i} \phi\left(\frac{|n_i|}{N}\right) e^{-itn_i^2 + i\theta_i n_i} \right)$$

where θ_i stands for $\begin{pmatrix} e^{i\theta_i} & \\ & e^{-i\theta_i} \end{pmatrix} \in SU(2)$, $1 \leq i \leq d$; $\theta_i \in \mathbf{T}$, $d+1 \leq i \leq d+e$.

Similar as before, we have the following dispersive estimate for $|\frac{t}{2\pi} - \frac{a}{q}| < \frac{1}{qN}$, $(a, q) = 1$

$$(5.8) \quad \|K(t, \cdot)\|_\infty \lesssim \left(\frac{N^3}{\sqrt{q}(1+N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \right)^d \left(\frac{N}{\sqrt{q}(1+N|\frac{t}{2\pi} - \frac{a}{q}|^{1/2})} \right)^e.$$

This implies

$$(5.9) \quad \|\rho(t)K(t, \cdot)\|_\infty \lesssim N^{\frac{5d}{2} + \frac{e}{2}}.$$

Then

(5.10)

$$\hat{\Lambda}_{Q,M}(n, m_1, \dots, m_d, n_{d+1}, \dots, n_{d+e}) = \lambda_{Q,M}(n, m_1, \dots, n_{d+1}, \dots) \cdot \text{Id}_{\prod_{i=1}^d (m_i+1) \times \prod_{i=1}^e (m_i+1)}$$

where

$$(5.11) \quad \lambda_{Q,M}(n, m_1, \dots, n_{d+1}, \dots) = \prod_{i=1}^d \phi\left(\frac{m_i(m_i+2)}{N^2}\right) \cdot \prod_{i=d+1}^{d+e} \phi\left(\frac{|n_i|}{N}\right)$$

$$(5.12) \quad \cdot \left[\left(\sum_{(a,q)=1, q \sim Q} \delta_{a/q} \right)^\wedge \cdot \hat{\omega}_{\frac{1}{NM}} - \alpha_{Q,M} \hat{\rho} \right] \left(n + \sum_{i=1}^d m_i(m_i+2) + \sum_{i=d+1}^{d+e} n_i^2 \right).$$

Then following the same lines as for $SU(2)$, we get the following proposition.

Proposition 5.13. *Let G be a rational product $SU(2)^d \times \mathbf{T}^e$. $F = P_N e^{it\Delta} f_0$, $f_0 \in L^2(G)$ with $\|f_0\|_2 = 1$. Let $m_\lambda = \{(t, x) \in \mathbf{T} \times G : |F(t, x)| > \lambda\}$. Let $p_0 = \frac{2(d+e+2)}{d+e}$. Then*

$$(i) m_\lambda \ll N^{\frac{(3d+e)p_0}{2} - (3d+e+2) + \epsilon} \lambda^{-p_0}, \text{ for } \lambda > CN^{\frac{5d}{4} + \frac{e}{4}}.$$

$$(ii) m_\lambda \lesssim N^{\frac{(3d+e)p}{2} - (3d+e+2)} \lambda^{-p}, \text{ for } \lambda > CN^{\frac{5d}{4} + \frac{e}{4}}, p > p_0.$$

Corollary 5.14. (i)

$$(5.15) \quad \|P_N e^{it\Delta} f\|_p \lesssim N^{\frac{3d+e}{2} - \frac{3d+e+2}{p}} \|f\|_2$$

holds for $p \geq 2 + \frac{8}{d+e}$.

(ii) If the above Strichartz estimate holds for $p > p_0 = \frac{2(d+e+2)}{d+e}$ with an N^ϵ loss, then it holds for $q > p$ without loss.

6. THE CASE OF THE COVERING GROUPS

We'd like to prove the following theorem.

Theorem 6.1. *Let $G \cong \mathbf{T}_1 \times \cdots \times \mathbf{T}_n \times K$, where T_i 's are circles and K a compact simply-connected semisimple Lie group. Let G be equipped with a “rational metric” g , which is defined to be $g = \otimes_{i=1}^n g_{\mathbf{T}_i} \otimes g_K$, where $g_{\mathbf{T}_i}, g_K$ are constant multiples of the canonical metric on \mathbf{T}_i, K respectively and periods of the linear Schrödinger flow on each component T_i, K are rational multiples of each other. Let d be the dimension of G and r the rank of G . Let $F = P_N e^{it\Delta} f_0$, $f_0 \in L^2(G)$ with $\|f_0\|_2 = 1$. Let $m_\lambda = |(t, x) \in \mathbf{T} \times G : |F(t, x)| > \lambda|$. Let $p_0 = \frac{2(r+2)}{r}$. Then*

(i) $m_\lambda \ll N^{\frac{dp_0}{2} - (d+2) + \epsilon} \lambda^{-p_0}$, for $\lambda > CN^{d - \frac{r}{2}}$.

(ii) $m_\lambda \lesssim N^{\frac{dp}{2} - (d+2)} \lambda^{-p}$, for $\lambda > CN^{d - \frac{r}{2}}$, $p > p_0$.

(iii)

$$\|P_N e^{it\Delta} f_0\|_p \lesssim N^{\frac{d}{2} - \frac{d+2}{p}}$$

holds for $p \geq 2 + \frac{8}{r}$.

(iv) If the above Strichartz estimate holds for $p > p_0$ with an N^ϵ loss, then it holds for $q > p$ without loss.

Slightly adapting the Tomas-Stein argument for $SU(2)^d \times \mathbf{T}^e$, we see that the above theorem follows from following proposition.

Proposition 6.2. *Let G be a simply-connected compact semisimple Lie group of dimension d and of rank r , $L = \{\sum_{i=1}^r n_i w_i, n_i \in \mathbf{Z}\}$ be the weight lattice for the root system of G , $\hat{G} = \{\sum_{i=1}^r n_i w_i \in L, n_i \geq 0\}$ the dominant integral weights, $\alpha \in P$ the positive roots, $\rho = \sum_{i=1}^r w_i = \frac{1}{2} \sum_{\alpha \in P} \alpha$, W the Weyl group. $D \in \mathbf{N}$ is chosen such that $\langle \lambda_1, \lambda_2 \rangle \in D^{-1} \mathbf{Z}$ for any $\lambda_1, \lambda_2 \in L$. Then the Schrödinger kernel*

(6.3)

$$K(t, x) = \sum_{\lambda \in \hat{G}} e^{-it(|\lambda + \rho|^2 - |\rho|^2)} \phi\left(\frac{|\lambda + \rho|^2}{N^2}\right) \frac{\prod_{\alpha \in P} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in P} \langle \alpha, \rho \rangle} \frac{\sum_{s \in W} \det(s) e^{i\langle s(\lambda + \rho), H \rangle}}{\sum_{s \in W} \det(s) e^{i\langle s(\rho), H \rangle}}$$

(with

$$(6.4) \quad K(t, 1) = \sum_{\lambda \in \hat{G}} e^{-it(|\lambda + \rho|^2 - |\rho|^2)} \phi\left(\frac{|\lambda + \rho|^2}{N^2}\right) \left(\frac{\prod_{\alpha \in P} \langle \alpha, \lambda + \rho \rangle}{\prod_{\alpha \in P} \langle \alpha, \rho \rangle}\right)^2$$

) satisfies the estimates

$$(6.5) \quad |K(t, x)| \lesssim \frac{N^d}{(\sqrt{q}(1 + N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^r}$$

for $|\frac{t}{2\pi D} - \frac{a}{q}| < \frac{1}{qN}$, $(a, q) = 1$. Here x is conjugate to $\exp H$.

Lemma 6.6. *Following notations of the previous lemma, we have*

(6.7)

$$(6.8) \quad \begin{aligned} K(t, x) &= \frac{e^{it|\rho|^2}}{(\prod_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle \lambda, H \rangle} \phi\left(\frac{|\lambda|^2}{N^2}\right) \prod_{\alpha \in P} \langle \alpha, \lambda \rangle \\ &= \frac{e^{it|\rho|^2}}{(\prod_{\alpha \in P} \langle \alpha, \rho \rangle) |W|} \sum_{\lambda \in L} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) \prod_{\alpha \in P} \langle \alpha, \lambda \rangle \frac{\sum_{s \in W} \det(s) e^{i\langle s(\lambda), H \rangle}}{\sum_{s \in W} \det(s) e^{i\langle s(\rho), H \rangle}} \end{aligned}$$

with

$$(6.9) \quad K(t, 1) = \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle)^2 |W|} \sum_{\lambda \in L} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) (\Pi_{\alpha \in P} \langle \alpha, \lambda \rangle)^2.$$

Here

$$(6.10) \quad \sigma(H) = \sum_{s \in W} \det(s) e^{i\langle s(\rho), H \rangle}$$

and $|W|$ is the cardinality of W .

Proof. The Weyl group W acts on the weight lattice L isometrically (preserving inner product), thus we have

$$\begin{aligned} \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle \lambda, H \rangle} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle &= \sum_{\lambda \in L} e^{-it|s(\lambda)|^2 + i\langle s(\lambda), H \rangle} \phi\left(\frac{|s(\lambda)|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, s(\lambda) \rangle \\ &= \det(s) \sum_{\lambda \in L} e^{-it|\lambda|^2 + i\langle s(\lambda), H \rangle} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle. \end{aligned}$$

Here we used

$$(6.11) \quad \Pi_{\alpha \in P} \langle \alpha, s(\lambda) \rangle = \det(s) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle.$$

Then the equality of (6.7) and (6.8) is obvious. Let $R = \hat{G} + \rho = \{\sum_{i=1}^r n_i w_i \in L, n_i \geq 1\}$, then using (6.11) we have

$$\begin{aligned} K(t, x) &= \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{\lambda \in D} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle \sum_{s \in W} \det(s) e^{i\langle s(\lambda), H \rangle} \\ &= \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{s \in W} \sum_{\lambda \in D} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, s(\lambda) \rangle e^{i\langle s(\lambda), H \rangle} \\ &= \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{s \in W} \sum_{\lambda \in D} e^{-it|s(\lambda)|^2} \phi\left(\frac{|s(\lambda)|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, s(\lambda) \rangle e^{i\langle s(\lambda), H \rangle} \\ &= \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{\lambda \in \cup_{s \in W} s(D)} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle e^{i\langle \lambda, H \rangle} \end{aligned}$$

For G being compact simply-connected semisimple, $\{\lambda \in L : \Pi_{\alpha} \langle \alpha, \lambda \rangle \neq 0\} = \cup_{s \in W} s(R)$ (disjoint union), which would imply

$$(6.12) \quad K(t, x) = \frac{e^{it|\rho|^2}}{(\Pi_{\alpha \in P} \langle \alpha, \rho \rangle) \sigma(H)} \sum_{\lambda \in L} e^{-it|\lambda|^2} \phi\left(\frac{|\lambda|^2}{N^2}\right) \Pi_{\alpha \in P} \langle \alpha, \lambda \rangle e^{i\langle \lambda, H \rangle}.$$

□

Lemma 6.13. *Following previous notations, let f be a function on the weight lattice $\Lambda = \{\sum_{i=1}^r n_i w_i, n_i \in \mathbf{Z}, 1 \leq i \leq l\}$, with $|D_{i_1} \cdots D_{i_n} f| \lesssim N^{A-n}$ uniformly in n , for some A . Here $D_i f := f(\cdots, n_i + 1, \cdots) - f(\cdots, n_i, \cdots)$. Let*

$$(6.14) \quad F = \sum_{\lambda \in \Lambda} e^{-it|\lambda|^2 + i\langle \lambda, H \rangle} \phi\left(\frac{|\lambda|^2}{N^2}\right) \cdot f.$$

Then for $|\frac{t}{2\pi D} - \frac{a}{q}| < \frac{1}{qN}$, $(a, q) = 1$, $q < N$, we have

$$(6.15) \quad |F| \lesssim \frac{N^{r+A}}{(\sqrt{q}(1 + N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^r}.$$

Proof. Write

$$\begin{aligned} |F|^2 &= \sum_{\lambda_1, \lambda_2 \in \Lambda} e^{-it(|\lambda_1|^2 - |\lambda_2|^2) + i\langle \lambda_1 - \lambda_2, H \rangle} \phi\left(\frac{|\lambda_1|^2}{N^2}\right) \phi\left(\frac{|\lambda_2|^2}{N^2}\right) f(\lambda_1) \bar{f}(\lambda_2) \\ &= \sum_{\mu = \lambda_1 - \lambda_2} e^{-it|\mu|^2 + i\langle \mu, H \rangle} \sum_{\lambda = \lambda_2} e^{-i2t\langle \mu, \lambda \rangle} \phi\left(\frac{|\mu + \lambda|^2}{N^2}\right) \phi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu + \lambda) \bar{f}(\lambda) \\ &\leq \sum_{|\mu| \leq CN} \left| \sum_{\lambda} e^{-i2t\langle \mu, \lambda \rangle} \phi\left(\frac{|\mu + \lambda|^2}{N^2}\right) \phi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu + \lambda) \bar{f}(\lambda) \right|. \end{aligned}$$

Now write $\lambda = \sum_{i=1}^r n_i w_i$, $g = \phi\left(\frac{|\mu + \lambda|^2}{N^2}\right) \phi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu + \lambda) \bar{f}(\lambda)$, then

$$\sum_{\lambda \in \Lambda} e^{-i2t\langle \mu, \lambda \rangle} g = \sum_{n_1, \dots, n_r \in \mathbf{Z}} (\prod_{i=1}^r e^{-itn_i \langle \mu, 2w_i \rangle}) g.$$

By summation by parts twice, we have

$$\sum_{n_1 \in \mathbf{Z}} e^{-itn_1 \langle \mu, 2w_1 \rangle} g = \left(\frac{e^{-it\langle \mu, 2w_1 \rangle}}{1 - e^{-it\langle \mu, 2w_1 \rangle}} \right)^2 \sum_{n_1 \in \mathbf{Z}} e^{-itn_1 \langle \mu, 2w_1 \rangle} D_1^2 g.$$

Here $(D_i g)(n_1, \dots, n_r) := g(\dots, n_i + 1, \dots) - g(\dots, n_i, \dots)$. Now do this summation by parts twice to n_1, \dots, n_r consequently only if $|1 - e^{-it\langle \mu, 2w_i \rangle}| \geq \frac{1}{N}$, and note that

$$|D_{i_1} \dots D_{i_n} g| \lesssim N^{2A-n}, \forall n$$

we obtain

$$\begin{aligned} & \left| \sum_{\lambda} e^{-i2t\langle \mu, \lambda \rangle} \phi\left(\frac{|\mu + \lambda|^2}{N^2}\right) \phi\left(\frac{|\lambda|^2}{N^2}\right) f(\mu + \lambda) \bar{f}(\lambda) \right| \\ & \lesssim N^{2A-r} \prod_{i=1}^r \frac{1}{(\max\{1 - e^{-it\langle \mu, 2w_i \rangle}, \frac{1}{N}\})^2} \\ & \lesssim N^{2A-r} \prod_{i=1}^r \frac{1}{(\max\{||t\langle \mu, 2w_i \rangle||, \frac{1}{N}\})^2}. \end{aligned}$$

Here $||\cdot||$ stands for the distance from 0 on a $[0, 2\pi)$ circle. Write $\mu = \sum_{j=1}^r m_j w_j$, we have

$$|F|^2 \lesssim N^{2A-r} \sum_{|m_j| \leq CN, \forall j} \prod_{i=1}^r \frac{1}{(\max\{||t \sum_{j=1}^r m_j \langle w_j, 2w_i \rangle||, \frac{1}{N}\})^2}.$$

Let $n_i = \sum_{j=1}^r m_j \langle w_j, 2w_i \rangle \cdot D$, where $D \in \mathbf{N}$ so that $\langle w_j, w_i \rangle \in D^{-1}\mathbf{Z}$. Note that the matrix $(\langle w_j, 2w_i \rangle D)_{i,j}$ is non-degenerate, we have

$$|F|^2 \lesssim N^{2A-r} \prod_{i=1}^r \left(\sum_{|n_i| \leq CN} \frac{1}{(\max\{||\frac{t}{D} n_i||, \frac{1}{N}\})^2} \right).$$

Then by a standard argument, we have

$$\sum_{n_i \leq C_N} \frac{1}{(\max\{|\frac{t}{D}n_i|, \frac{1}{N}\})^2} \lesssim \frac{N^3}{(\sqrt{q}(1+N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^2},$$

we finally obtain

$$|F|^2 \lesssim \frac{N^{2A+2r}}{(\sqrt{q}(1+N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^{2r}}.$$

□

Apply the above lemma to the Schrödinger kernel $K(t, x)$, we get the following.

Lemma 6.16. *Following notations of the previous lemmas, we have*

$$(6.17) \quad |K(t, x)| \lesssim \frac{1}{|\sigma(H)|} \frac{N^{\frac{d+r}{2}}}{(\sqrt{q}(1+N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^r}$$

and

$$(6.18) \quad |K(t, 1)| \lesssim \frac{N^d}{(\sqrt{q}(1+N|\frac{t}{2\pi D} - \frac{a}{q}|^{1/2}))^r}.$$

Thus Lemma 6.2 is established for $x = 1$ or $|\sigma(H)| \gtrsim N^{-\frac{d-r}{2}}$. Since

$$(6.19) \quad \sigma(H) = e^{-i\langle \rho, H \rangle} \prod_{\alpha \in P} (e^{i\langle \alpha, H \rangle} - 1),$$

in particular we have that Lemma 6.2 is established for all H with $|\langle \alpha, H \rangle| \gtrsim N^{-1}$, $\forall \alpha \in P$. Here $\|\cdot\|$ stands for the distance from 0 on the $[0, 2\pi)$ circle.

For the case when $|\langle \alpha, H \rangle| \ll N^{-1}$ for some $\alpha \in P$, we have the following proposition.

Proposition 6.20. *Lemma 6.2 still holds when $|\langle \alpha, H \rangle| \ll N^{-1}$ for some $\alpha \in P$.*

Proof. The idea is to pick up the α' 's with $|\langle \alpha, H \rangle| \ll N^{-1}$, $\alpha \in P$, to form a sub root system. Consider $P_0 = \{\alpha_0 \in P : |\langle \alpha_0, H \rangle| \leq 10^{-6}N^{-1}\}$. Then consider all $\alpha \in P$ that are linear combinations of elements in P_0 , which form a sub positive root system, denoted P' . Thus $\alpha \in P'$ satisfies $|\langle \alpha, H \rangle| \leq 10^{-5}N^{-1}$. Consider the subgroup of the Weyl group W generated by P' , denoted W' , which is the Weyl group corresponding to P' . Now rewrite the character as follows

$$\begin{aligned} \chi_j &= \frac{\sum_{s \in W} \det s \sum_{s' \in W'} \det s' e^{i\langle s' s(\lambda), H \rangle}}{\prod_{\alpha \notin P'} (e^{i\langle \alpha, H \rangle} - 1) \cdot \prod_{\alpha \in P'} (e^{i\langle \alpha, H \rangle} - 1)} \\ &= \sum_{s \in W} \frac{\det s \cdot e^{i\langle s(\lambda), H_{\perp} \rangle}}{\prod_{\alpha \notin P'} (e^{i\langle \alpha, H \rangle} - 1)} \cdot \underbrace{\frac{\sum_{s' \in W'} \det s' e^{i\langle s' s(\lambda), H_{\parallel} \rangle}}{\prod_{\alpha \in P'} (e^{i\langle \alpha, H_{\parallel} \rangle} - 1)}}_g. \end{aligned}$$

Here H_{\parallel} is the projection of H onto $\text{Span}(P')$ and H_{\perp} is its orthogonal complement.

Now g is a Weyl character corresponding to the root system P' . By derivative estimates (2.10), that is

$$|D_{i_1} \cdots D_{i_n} g| \lesssim N^{p'-n}$$

where p' is the cardinality of P' , we can absorb g into f in (6.14) and perform the exponential sum estimates. This would finish the proof of Lemma 6.2, noting that $|e^{i\langle \alpha, H \rangle} - 1| \gtrsim N^{-1}$ for $\alpha \notin P'$. □

7. THE CASE OF COMPACT LIE GROUPS

Lastly we upgrade the Strichartz estimates on the covering groups to those on the original compact Lie groups.

Theorem 7.1. *(Theorem 2.7 in [3]) Let H be a closed subgroup of a Lie group G . Then there is a unique manifold structure on the quotient space G/H such that the projection map $\pi : G \rightarrow G/H$ is a smooth submersion.*

Moreover, given a biinvariant metric on G , the projection π induces on G/H a Riemannian metric such that the Laplace-Beltrami operator on $C^\infty(G/H)$ is identified with the Laplace-Beltrami operator on

$$(7.2) \quad C_{inv}^\infty(G) := \{f \in C^\infty(G) \text{ such that } f(x) = f(xg), \forall x \in G, g \in H\}$$

and the diagram commutes:

$$\begin{array}{ccc} C^\infty(G/H) & \xrightarrow{\pi^*} & C_{inv}^\infty(G) \\ \Delta_{G/H} \downarrow & & \downarrow \Delta_G \\ C^\infty(G/H) & \xrightarrow{\pi^*} & C_{inv}^\infty(G). \end{array}$$

This theorem implies that the following diagram commutes:

$$\begin{array}{ccc} L^2(G/H) & \xrightarrow{\pi^*} & L_{inv}^2(G) \\ e^{it\Delta_{G/H}} \downarrow & & \downarrow e^{it\Delta_G} \\ L^2(G/H) & \xrightarrow{\pi^*} & L_{inv}^2(G). \end{array}$$

Applying this for the case when H is finite and central, noting the isometry

$$L^p(G/H) \xrightarrow{\pi^*} L_{inv}^p(G), H^s(G/H) \xrightarrow{\pi^*} H_{inv}^s(G),$$

we have the following proposition.

Proposition 7.3. *Let G be a compact Lie group, H is finite and central. Then the Strichartz estimates for G/H*

$$(7.4) \quad \|e^{it\Delta_{G/H}} f\|_{L_t^p(I, L_x^r(G/H))} \lesssim \|f\|_{H^s(G/H)}$$

is equivalent to

$$(7.5) \quad \|e^{it\Delta_G} f\|_{L_t^p(I, L_x^r(G))} \lesssim \|f\|_{H^s(G)}$$

uniformly for all $f \in H_{inv}^s(G)$. In particular, any Strichartz inequality on G passes to that on G/H .

Applying this proposition to Theorem 6.1, the proof of Theorem 1.1 is finished.

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